Berry's phase, chaos, and the deformations of Riemann surfaces

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Parametrized families of Landau Hamiltonians on compact Riemann surfaces corresponding to classically chaotic families of geodesic motion are investigated. The parameters describe deformations of such surfaces with genus $g \geqslant 1$. It is shown that the adiabatic curvature (responsible for Berry's phase) of the lowest Landau level for $g>1$ is the sum of two terms. The first term is proportional to the natural symplectic form on deformation space, and the second is a fluctuating term reflecting the chaos of the geodesic motion for $g>1$. For $g=1$ (integrable motion on the torus) we have no fluctuating term. We propose our result to be interpreted as a curvature analog of the well-known semiclassical trace formulas. Connections with the viscosity properties of quantum Hall fluids on such surfaces are also pointed out. An interesting possibility in this respect is the fractional quantization of certain components of the viscosity tensor of such fluids. [S1063-651X(97)09410-5]

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Parametrized families of quantum systems exhibit interesting anholonomy properties with a wide range of physical applications [1]. It is a particularly interesting case when the family of quantum systems corresponds to a family of classically chaotic systems [2]. An interesting question concerning such systems is the following: how does the chaos of the underlying classical system manifest itself in the anholonomy properties of eigensubspaces of the corresponding Hamiltonian? An example illustrating this question can be obtained by quantizing the chaotic geodesic motion on a compact surface of constant negative curvature under the influence of a constant magnetic field [3,4]. This means that we have to consider the spectral properties of Landau Hamiltonians on a compact Riemann surface of genus $g>1$. Parametrized families of such Hamiltonians have been considered in [5], where the parameter space was spanned by $2g$ Aharonov-Bohm fluxes. The adiabatic curvature responsible for the anholonomy in this case can be related to the charge transport coefficients on this multiply connected surface. The remarkable result in [5] was the fact that the curvature splits into an integral (i.e., a quantized) part and a fluctuating part. In this spirit, in this paper we consider a more general case: we deform the surface itself by employing the so-called Teichmüller deformations [6] to obtain a family of Landau Hamiltonians parametrized by $3g-3$ complex parameters corresponding to a family of classically chaotic systems. In order to give an answer to the question posed above we calculate the adiabatic curvature of the lowest Landau level. The curvature in this case can be related to the components of the viscosity tensor for a quantum-Hall fluid with an energy gap at zero temperature [7]. Since the theoretical tools employed here are well known from string theory (see, e.g., [8]) we merely outline the basic ideas.

To begin with and to gain some insight we first recall and slightly reformulate results concerning the torus $g=1$ case [7,9]. The Teichmüller space $\mathcal{T}_1$ is just the upper half plane. The family of tori Σ_1 is parametrized by a complex parameter $\tau=\tau_1+i\tau_2$ with positive imaginary part. It is represented by parallelograms with opposite sides identified, with lattice vectors $1/\sqrt{\tau_2}$, and $\tau/\sqrt{\tau_2}$ normalized to preserve the (unit) area under deformations. Such deformations of tori [with coordinates (x,y)] can be represented as the deformations of the metric. Using the coordinates θ_1,θ_2 with the property $z=x+iy=(1/2\pi\sqrt{\tau_2})(\theta_1+\theta_2\tau)$, the metric is defined by $\hat{g}=dx^2+dy^2\equiv g_{ij}(\tau,\overline{\tau})d\theta^i d\theta^j$. Our Landau Hamiltonian is $H_B=-\frac{1}{2}[(\partial_x-iA_x)^2+(\partial_y-iA_y)^2]$ with $F=B\,dx\wedge dy=dA$. The flux of the constant magnetic field B has to be quantized, $(1/2\pi)\int_{\Sigma_1}F=N$, i.e., $N=B/2\pi\in\mathbf{Z}$. Using the metric $g_{ij}(\tau,\overline{\tau})$ adapted to the deformed tori the Landau Hamiltonians $H_B(\tau,\overline{\tau})=-\frac{1}{2}g^{ij}(\tau,\overline{\tau})D_iD_j$ can be related to suitably chosen operators such as $H_B(\tau,\overline{\tau})-B/2\sim D_B^{\dagger}(\overline{\tau})D_B(\tau)$, i.e., the operator D_B exhibits a holomorphic dependence on the complex deformation parameter. The lowest Landau level is N-fold degenerate, and the states $\psi_\alpha,\alpha=1,\ldots,N$, can be obtained as the zero modes of D_B. The unnormalized states can be expressed in terms of the theta functions with characteristics and they also depend holomorphically on τ. Using the normalization properties of the theta functions we have $\mathcal{N}_{\alpha\beta}\equiv\langle\psi_\alpha|\psi_\beta\rangle=(\tau_2)^{-1/2}\delta_{\alpha\beta}$. The adiabatic curvature for the spectral projector P of the ground state is defined as $\omega(P)=\mathrm{Tr}(PdP\wedge dP)$. By defining the exterior derivatives $\partial\equiv d\tau\partial/\partial\tau$, and $\overline{\partial}\equiv d\overline{\tau}\partial/\partial\overline{\tau}$ $d\equiv\partial+\overline{\partial}$, one can prove the relation

$$\omega(P)=i\partial\overline{\partial}\ln\det\mathcal{N}_{\alpha\beta}\tag{1}$$

valid for an arbitrary wave function with holomorphic dependence on some complex parameters. In our case we have the rank-N projector P, hence we obtain the result

$$\omega(P)=-\frac{B}{8\pi}\frac{d\tau_1\wedge d\tau_2}{{\tau_2}^2}\equiv-\frac{B}{4\pi}\omega_{\mathrm{WP}},\tag{2}$$

where ω_{WP} is the Weil-Petersson two-form [10]. In the length-twist coordinates $l=1/\sqrt{\tau_2}$ and $t=-\tau_1/\sqrt{\tau_2}$, the allowed deformations are executed by cutting the torus along the geodesic with length l twisting by an angle $2\pi t/l$ and then gluing the pieces together. In these coordinates $\omega_{\mathrm{WP}}=dt\wedge dl$, hence it is the natural symplectic structure. Notice also that the gauge structure is Abelian since $\mathcal{N}_{\alpha\beta}$ is diagonal, yielding the formula $\omega(P)=\Sigma_1^N \mathcal{F}_{\alpha\alpha}$ where $\mathcal{F}_{\alpha\beta}=\mathcal{F}\delta_{\alpha\beta}$. Here $\mathcal{F}$ is the two-form whose flux through a surface bounded by the closed curve C gives Berry's phase. Hence it follows that each individual level cycled adiabatically along C picks up a Berry phase $e^{i\gamma_C}$, where $\gamma_C=-\frac{1}{2}(\text{Area})$ with "Area" being the area enclosed by C with respect to the symplectic form ω_{WP}. Note that a calculation for the excited states labeled by $n=1,2\ldots$ can also be carried out with the result $\gamma_C^{(n)}=-(n+1/2)(\text{Area})$ [9]. Recall also that ω_{WP} is invariant with respect to the modular group $\Gamma_1\equiv\text{PSL}(2,\mathbf{Z})$, then it also defines a two-form on moduli space $\mathcal{M}_1\equiv\mathcal{T}_1/\Gamma_1$, which is a sphere with one puncture and two orbifold points. Hence the true parameter space for the deformations of tori is the moduli space. On $\mathcal{M}_1$ we have also non-Abelian anholonomy defined by some projective unitary representation of Γ_1 arising from the transformation properties of the theta functions [9]. Moreover, we can also allow deformations of our torus for wich the geodesic length l is getting pinched down to zero. The torus in this way has developed a node. With the help of such deformations we can *compactify* $\mathcal{M}_1$. We can extend $\omega(P)$ to the resulting compact space $\overline{\mathcal{M}}_1$, which is an orbifold. One can now characterize the nondissipative part of the viscosity of a quantum Hall fluid on the torus with a full Landau level and an energy gap at zero temperature by $-(N/2)\omega_{\text{WP}}$ living on $\overline{\mathcal{M}}_1$. We can also calculate $(1/2\pi)\int_{\overline{\mathcal{M}}_1}\omega(P)$ yielding the rational number $-N/24$, which is a topological invariant for the eigenstate bundle of the lowest Landau level. This topological Chern number is completely analogous to the one found for the conductance in the theory of the quantum Hall effect.

Now we turn to the $g\geqslant 2$ case. Locally we can choose isothermal coordinates in which the metric takes the form $\hat{g}=e^{2\sigma}dzd\overline{z}=2g_{z\overline{z}}dzd\overline{z}$. The deformations of our surfaces will be described by suitable deformations of the metric. In analogy with the torus case we will be interested in the Teichmüller deformations. The Teichmüller space can be described as $\mathcal{T}_g=\text{Metr}_{\text{const}}(\Sigma_g)/\text{Diff}_0(\Sigma_g)$. We define elements of $\text{Metr}_{\text{const}}(\Sigma_g)$ by metrics with scalar curvature $R_{\hat{g}}=-1$, i.e., $e^{2\sigma}=\triangle\sigma$. By the Gauss-Bonnet theorem we see that we are considering those area-preserving metric deformations that *cannot be compensated* by diffeomorphisms connected to the identity (i.e., infinitesimal reparametrizations) . It is well known that the metric deformations near the metric $\hat{g}$ we are interested in are of the form [11]

$$\hat{g}(\tau)=e^{2\sigma}|dz+g^{z\overline{z}}\overline{\tau^j}\overline{f_{jzz}}d\overline{z}|^2,\quad j=1,\ldots,3g-3,\tag{3}$$

where the f_{izz} form a basis in the $3g-3$ complex dimensional space of *holomorphic quadratic differentials*, i.e., the ones transforming as the components of a quantity $f_{zz}dz^2$ with the property $\partial_{\overline{z}}f=0$. The complex coordinates τ^j now coordinates for $\mathcal{T}_g$ in the vicinity of $\hat{g}$.

Having described the deformations we are interested in, now we turn to the description of Landau Hamiltonians on Σ_g, and their deformations. Our Landau Hamiltonian is $H_B=-\frac{1}{2}e^{-2\sigma}[(\partial_x-iA_x)^2+(\partial_y-iA_y)^2]$, with $F=dA=Be^{2\sigma}dx\wedge dy$, $B=\text{const}$. The flux on Σ_g has to be quantized; combined with the Gauss-Bonnet theorem this implies that B must be rational. ($\hbar=e=1$). However, further arguments [4] show that a noninteger B can only be introduced consistently provided we also introduce fluxes through the handles. Hence in the absence of fluxes we have $B\in\mathbf{Z}$. The next step is to introduce complex coordinates, and the gauge choice $A=A_zdz+A_{\overline{z}}d\overline{z}=-2iB(\partial_z\sigma)dz$, to write H_B as

$$\begin{aligned}H_B&=-2e^{-2\sigma}[\partial_z\partial_{\overline{z}}-2B(\partial_z\sigma)\partial_{\overline{z}}]+\frac{B}{2}\\&=D_B^\dagger D_B+\frac{B}{2}=D_{B+1}D_{B+1}^\dagger-\frac{B}{2},\end{aligned}\tag{4}$$

where $D_B=2e^{-2\sigma}\partial_{\overline{z}}$ and $D_B^\dagger=-\partial_z+2(B-1)(\partial_z\sigma)$, which are just the operators introduced in [8]. Since H_B is positive definite we have for $B>0$ $\text{Ker}D_{B+1}^\dagger=0$. Using now the Riemann-Roch theorem [6] we obtain the degeneracy of the lowest Landau level as $\dim\text{Ker}D_B=(2B-1)(g-1)$. Hence we have $H_B|\psi_\alpha\rangle=(B/2)|\psi_\alpha\rangle$ $\alpha=1,\ldots,N-g+1$ for $N>2g-2$ ($B>1$) the only case considered here, where $N=(1/2\pi)\int_{\Sigma_g}F$ is the number of flux quanta. The next step is to deform the operators D_B and $D_B^\dagger$ by using the (3) metric deformation. Here we meet a pleasant surprise: D_B ($D_B^\dagger$) varies holomorphically (antiholomorphically) with τ^j. The same property is valid for $|\psi_\alpha\rangle$ ($\langle\psi_\alpha|$). In order to calculate the adiabatic curvature we calculate the quantity $i\partial\overline{\partial}\ln\det{}'D_B^\dagger(\overline{\tau})D_B(\tau)$ (recall our convention $\partial\equiv d\tau^j\partial/\partial\tau^j$). We define this determinant via zeta function regularization, and the prime indicates deletion of zero modes. Using the formula [12] valid for parametrized families of operators K,

$$\overline{\partial}\ln\det{}'K=-\lim_{t\to 0}\text{Tr}[(1-P)\overline{\partial}K(K)^{-1}e^{-tK}],\tag{5}$$

straightforward calculation shows that ($\text{Ker}D^\dagger=0$ for $B>1$)

$$\begin{aligned}i\partial\overline{\partial}\ln\frac{\det{}'D_B^\dagger D_B}{\det\langle\psi_\alpha|\psi_\beta\rangle}=i\lim_{t\to 0}t\int_0^1du\ \text{Tr}(\overline{\partial D}^\dagger e^{-tuDD^\dagger}\\ \times\partial De^{-(1-u)tD^\dagger D}).\end{aligned}\tag{6}$$

Using the explicit forms for the deformations of D and $D^\dagger$ and short time Heat-kernel techniques [6,8] (see also [13] in this respect) we obtain the formula

$$i\frac{C_B}{6\pi}\left(\int e^{-4\sigma}[\partial_{\overline{z}}f_j\partial_z\overline{f_k}+\tfrac{1}{2}f_j\overline{f_k}\triangle\sigma]dzd\overline{z}\right)d\tau^j\wedge d\overline{\tau^k}\tag{7}$$

for the right-hand side of Eq. (6), where $C_B=6B^2-6B+1$. For the allowed deformations $\partial_{\overline{z}}f=0$ and $\Delta\sigma=e^{-2\sigma}$, hence using the definition of the Weil-Petersson two-form [10] and Eq. (1) we can write down our main result:

$$\omega(P)=-\frac{1}{12\pi}C_B\omega_{\text{WP}}+i\partial\overline{\partial}\ln Z(B),\quad B\geqslant 2,\tag{8}$$

where we also expressed $\ln\det{}'D_B^\dagger D_B$ in terms of Selberg's zeta function [14], which is defined as

$$Z(B)=\prod_p \prod_{k=0}^{\infty} (1-e^{-(B+k)l(p)}). \quad (9)$$

Note that in this definition of $Z(B)$ our Riemann surface Σ_g is uniformized as $\mathcal{H}/\Gamma$ where $\mathcal{H}$ is the upper half plane and Γ is a discrete subgroup of PSL(2,**R**). Here p are inconjugate primitive hyperbolic elements of Γ representing simple closed geodesics with hyperbolic length $l(p)$.

We can decompose our surface Σ_g into ''pants'' by cutting it along $3g-3$ simple closed geodesics [10]. Denoting the lengths by l_j and the twist parameters t_i related to the angle twists by $\theta_i=2\pi t_i/l_i$ (Frenchel-Nielssen coordinates), we can express $\omega_{\rm WP}$ using Scott Wolpert's formula [10] as $\omega_{\rm WP}=\Sigma_{i=1}^{3g-3}dt^i\wedge dl^i$, which defines an integrable Hamiltonian system on the ''space of shapes'' $\mathcal{T}_g$. $\omega_{\rm WP}$ is Kähler on $\mathcal{T}_g$ [10] hence it is closed, and since on $\mathcal{T}_g$ there are *globally* defined analytic coordinates [10] there is a globally defined Kähler potential $U_{\rm WP}$ for which $\omega_{\rm WP}=i\partial\bar{\partial}U_{\rm WP}$. Moreover $\omega_{\rm WP}$ is invariant with respect to the mapping class group of the surface $\Gamma_g={\rm Diff}^+(\Sigma_g)/{\rm Diff}_0(\Sigma_g)$, hence it also defines a two-form on moduli space $\mathcal{M}_g=\mathcal{T}_g/\Gamma_g$. Moreover, by attaching a boundary to $\mathcal{T}_g$ (see below) we can also compactify $\mathcal{M}_g$, obtaining $\overline{\mathcal{M}_g}$ on which $[\omega_{\rm WP}]/\pi^2$ defines a rational cohomology class [10]. Hence by integrating $(1/2\pi)\omega(P)$ on two-cycles the first term of Eq. (8) will yield rational numbers. It is a topological invariant (''monopole charge'') in the orbifold sense. Hence in the components of the viscosity tensor the first term of Eq. (8) yields certain rational numbers depending on the choice of B and the choice of two-cycle representing the deformation in question. As an example of this ''rational quantization'' we consider a surface with genus g consisting of two surfaces of $g=1$ and $g-1$ joined together by a single point: a node. The first part is just a torus punctured only once. By merely allowing deformations of this ''leaky torus'' we can define a two-cycle $\mathcal{C}$ in $\overline{\mathcal{M}_g}$. One can prove that $\omega_{\rm WP}|_{\mathcal{C}}=\omega_{\rm WP}|_{\overline{\mathcal{M}_{1,1}}}$ where $\overline{\mathcal{M}_{1,1}}$ is just the moduli space of the leaky torus. Using the result [15] $\int_{\overline{\mathcal{M}_{1,1}}}\omega_{\rm WP}=\pi^2/6$ one can prove that the *first part* of $(1/2\pi)\omega(P)$ integrated over $\mathcal{C}$ is $-\frac{1}{24}(B^2-B+\frac{1}{6})$ $(B\in\mathbf{Z},\ B\geqslant 2)$.

In the second term of Eq. (8) ln$Z(B)$ is playing the role of the Kähler potential. This term describes fluctuations in the components of the viscosity tensor of a quantum Hall fluid on Σ_g [7]. Its geometrical meaning is also clear: it reflects the change in the geodesic length spectrum when applying Frenchel-Nielssen deformations to our surface. According to Eq. (2) for the torus $(g=1)$ we have no fluctuating term, in accordance with the fact that the geodesic motion on Σ_1 is integrable.

From the physical point of view it is important to allow also those deformations of surfaces where the length of one (or more) of its geodesics is pinching to zero. The inclusion of such degenerate surfaces (i.e., surfaces with nodes) amounts to attaching a boundary to Teichmüller space obtaining $\overline{\mathcal{T}_g}$. In this case we have to consider the behavior of $\omega(P)$ as one of its geodesic lengths l_0 goes to zero. Consider first the extension of $\omega_{\rm WP}$ to $\overline{\mathcal{T}_g}$ [16]. In a suitable complex coordinate u (related to the length-twist parameters as $l_0\sim 1/\ln|u|^{-1}$ and t_0 can be written as l_0 times the phase of u), the degenerating part of $\omega_{\rm WP}$ is $du\wedge d\bar{u}/(\ln|u|^{-1})^3|u|^2$. Hence the extension is not smooth. Even though $\omega_{\rm WP}$ has singularities, it is closed (viewed as a current, i.e., $\int\omega\wedge d\varrho=0$, for some $2n-3$ form ϱ with compact support, $n=\dim_{\mathbf{C}}\overline{\mathcal{T}_g}$) [17]. For the behavior of the second term in Eq. (8) we have to consider the asymptotic behavior of Selberg's zeta function as $l_0\rightarrow 0$. Using the property $\eta(-1/\tau)=(-i\tau)^{1/2}\eta(\tau)$ with $\tau\equiv il_0/2\pi$ of Dedekind's eta function it can be shown that $Z(B)\sim l_0^{-B+1/2}e^{-\pi^2/6l_0}$. Hence the part ln$Z(B)$ of the Kähler potential is also singular. This shows that $\omega(P)$ is closed and *locally* in the sense of distributions it can be expressed as $\omega(P)=i\partial\bar{\partial}V$ for some V continuous.

Concerning the problem of Berry phases, due to the existence of *globally* defined analytic coordinates, we expect the matrix $\mathcal{N}_{\alpha\beta}$ to be globally diagonalizable over $\overline{\mathcal{T}_g}$ [see Eq. (1) and the torus case in this respect]. In this case the gauge structure would be Abelian and we would have an expression for $\mathcal{F}$ (the Robbins-Berry two-form [2]) as Eq. (8) divided by $(2B-1)(g-1)$. In this case we would have a two-form that is closed, and exhibits singularities. Moreover, we would then define the gauge potential $\mathcal{A}\equiv iJdV$ with $(2B-1)(g-1)V=-(1/12\pi)C_BU_{\rm WP}+\ln Z(B)$ and J is the complex structure on $\overline{\mathcal{T}_g}$. Moreover, in [2] the classical limit of $\mathcal{F}$ was considered and a convergent formula valid also for chaotic systems was obtained. The authors also derived semiclassical corrections to it associated with classical periodic orbits. In this spirit it is tempting to interpret Eq. (7) as the ''curvature analogue'' of the well-known trace formulas. On the left-hand side there is a quantity that is quantum in origin, it is Tr$(PdP\wedge dP)$. On the right-hand side there are quantities expressed in terms of the symplectic structure in length-twist parameters (the term with $\omega_{\rm WP}$) and the second variation of the length spectrum with respect to these parameters [the term with $Z(B)$]. We might also conjecture that the semiclassical formula in [2] is exact for our example of surfaces with constant negative curvature. These results might show the way to generalize the Hannay angles [18] (the classical analogs of Berry's phase for integrable systems) for chaotic systems. In order to prove these conjectures however, further investigations are needed.

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